Fermi-liquid theory

The Hamiltonian of interacting electrons

$$\hat{H}_0 = \sum_{\vec{p},\sigma} \xi(p)\, \hat{a}^+_{p\sigma} \hat{a}_{p\sigma} + \frac{1}{2} \sum_{p,k,q} U_q\, \hat{a}^+_{p+q\sigma} \hat{a}^+_{k-q\sigma'} \hat{a}_{k\sigma'} \hat{a}_{p\sigma}$$

The task of the Fermi-liquid theory is to find the character of excitations in the ground state of the system.

Depending on the character of the interactions, possible ground states may include:

- Superconductivity
- Ferromagnetism
- Charge density wave
- Spin waves

⋮

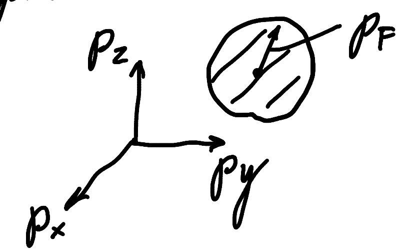
However, most electronic systems behave like systems of weakly interacting fermions (quasiparticles)

= have, e.g., the same temperature dependencies of heat capacity, conductivity, etc.

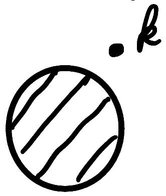
Landau's phenomenological theory

When $T=0$ and the interaction $U=0$, the electrons fill a sphere of radius

$$p_F = \left(3\pi^2 \frac{N}{V}\right)^{\frac{1}{3}}$$



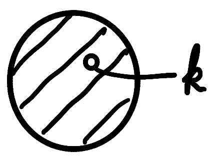
Elementary excitations



$$\xi(\vec{p}) = \frac{p^2}{2m} - \frac{p_F^2}{2m} \approx$$

$$\approx v_F (p - p_F)$$

$$v_F \approx \frac{p_F}{m}$$ (moved an electron from the Fermi surface to outside of the surface)



$$\xi(\vec{p}) = \frac{p_F^2}{2m} - \frac{p^2}{2m} \approx$$

$$\approx v_F (p_F - p)$$

for $|p_F - p| \ll p_F$

$$\boxed{\xi_{\vec{p}} \approx v_F |p_F - p|}$$

Now let's „switch on" the interactions. Then the ground state of the non-interacting electron gas transforms adiabatically into the ground state of the interacting system.

The excitations transform into the excitations of the interacting system = quasiparticles

The quasiparticles are characterised by momenta $\vec{p}$

We may introduce the density $n(\vec{r}, \vec{p})$ of the quasiparticles = the density of quasiparticles in momentum space divided by $(2\pi\hbar)^3$

If $n(\vec{r}, \vec{p})$ deviates slightly from that of the ground state, May be used as a definition of n

$$\delta E = V \int \varepsilon(\vec{p})\, \delta n(\vec{p}, \vec{r}) \frac{d\vec{p}\, d\vec{r}}{(2\pi\hbar)^3}$$

ground state,

$$\delta E = V \int \varepsilon(\vec{p})\, \delta n(\vec{p},\vec{r})\, \frac{d\vec{p}\, d\vec{r}}{(2\pi\hbar)^3}$$

$\varepsilon(\vec{p})$ plays the role of the quasiparticle dispersion but the total energy E is not given by the integral of $\varepsilon(\vec{p})$ over all states because of interactions.

Assume for simplicity $\delta n(\vec{p},\vec{r})$ is homogeneous in space (in quasiclassics, the system may be split into homogeneous regions anyway) and introduce spin

$$\delta E = \sum_{\vec{p},\sigma} \varepsilon(\vec{p},\sigma)\,\delta n(\vec{p},\sigma) + \frac{1}{2V} \sum_{\substack{\vec{p},\sigma \\ \vec{p}',\sigma'}} f(\vec{p},\sigma;\vec{p}',\sigma')\,\delta n(\vec{p},\sigma)\,\delta n(\vec{p}',\sigma')$$

The continuous version

$$\varepsilon'(\vec{p},\sigma) = \varepsilon(\vec{p},\sigma) + \int f(\vec{p},\sigma;\vec{p}',\sigma')\,\delta n(\vec{p}',\sigma')\,\frac{d\vec{p}'}{(2\pi\hbar)^3}$$

By symmetry, f must have the form

$$f(\vec{p},\sigma;\vec{p}',\sigma') = f_s(\cos\theta) + f_a(\cos\theta)\,\hat{\vec{\sigma}}\cdot\hat{\vec{\sigma}}'$$

Angle

(vanishes for an ideal gas)

The fact that states are adiabatically connected to those of the non-interacting system means the entropy is given by

$$S = -\sum_{\vec{p}\sigma} \left[n_{\vec{p}\sigma} \ln n_{\vec{p}\sigma} + (1-n_{\vec{p}\sigma}) \ln(1-n_{\vec{p}\sigma}) \right]$$

Consider $\Omega = E - TS - \mu N$ as a function with fixed μ and T and minimise wrt $n_{\vec{p}\sigma}$

$$\delta\Omega = \delta E - T\delta S - \mu \delta N =$$

$$= \sum_{\vec{p}\sigma} \varepsilon(\vec{p},\sigma)\,\delta n_{\vec{p}\sigma} - T\sum_{\vec{p},\sigma} \delta n_{\vec{p}\sigma}\left(\ln n_{\vec{p}\sigma} - \ln(1-n_{\vec{p}\sigma}) + 1 - 1\right)$$

$$- \mu \sum_{\vec{p},\sigma} \delta n_{\vec{p}\sigma}$$

Coeffs. before each $\delta n_{\vec{p}\sigma}$ vanish $\rightarrow$

$$\ln \frac{\bar{n}_{\vec{p}\sigma}}{1-\bar{n}_{\vec{p}\sigma}} = -\frac{1}{T}\left[\varepsilon(\vec{p},\sigma) - \mu\right]$$

$$\rightarrow \quad \boxed{\bar{n}_{\vec{p}\sigma} = \frac{1}{e^{\beta[\varepsilon(\vec{p},\sigma)-\mu]}+1}}$$

$$\varepsilon(p) \approx \varepsilon_F + (p-p_F)\underbrace{\left(\frac{\partial \varepsilon}{\partial p}\right)_{FS}}_{= v_F} + O\left((p-p_F)^2\right)$$

v_F - Fermi velocity (may be quite different from that in a non-interacting system)

The <u>effective mass</u> $m^* = \dfrac{p_F}{v_F} = \dfrac{p_F}{\left(\frac{\partial\varepsilon}{\partial p}\right)_{FS}}$

The heat capacity $C_V = \dfrac{\pi^2}{3} T\, \nu(\varepsilon_F)$

- uses only that the DoS is constant near the Fermi surface $\left(\nu(\varepsilon_F) = \dfrac{p_F^2}{\pi^2 v_F}\right)$

$$\boxed{\frac{(C_V)_{real}}{(C_V)_{ideal}} = \frac{m^*}{m}}$$

Kinetic equation. Collective modes

$$n = n(\vec p, \vec r, t)$$

$$\frac{\partial n}{\partial t} + \frac{\partial n}{\partial \vec r}\frac{\partial \varepsilon}{\partial \vec p} - \frac{\partial n}{\partial \vec p}\frac{\partial \varepsilon}{\partial \vec r} = 0$$

For small deviations of n from that of the ground state,

$$\frac{\partial \delta n}{\partial t} + \frac{\partial \delta n}{\partial \vec r}\frac{\partial \varepsilon^{(0)}}{\partial \vec p} - \frac{\partial \delta\varepsilon}{\partial \vec r}\frac{\partial n_0}{\partial \vec p} = 0$$

All solutions are expected to be very close to the Fermi surface, $\delta n \propto \delta(|\vec p| - p_F)$, which is why it is convenient to introduce the variable

$$u(\vec n) = \int \delta n(\vec p)\, d|\vec p|$$

(depends on the direction $\vec n$ on the Fermi surface)

Seek solutions in the form $u \propto e^{-i\omega t + i\vec k\vec r}$

Then from the previous kinetic equation

$$\boxed{(\vec\omega - \vec k\cdot \underset{\overset{|||}{v_F\vec n}}{\vec v})\, \vec u(\vec n) = \vec k\vec v \int F(\vec n, \vec n')\, u(\vec n')\, \frac{d\vec n'}{4\pi}}$$

where $F(\vec n, \vec n') = \nu_0\, f(\vec p, \vec p')\Big|_{|\vec p| = |\vec p'| = p_F}$

DoS for one spin projection

To derive the rhs part we use that

$$\delta\varepsilon = \int f(\vec p, \vec p')\, \delta n(\vec p', r)\, \frac{d\vec p'}{(2\pi\hbar)^3}$$

$$\delta\varepsilon = \int f(\vec{p}, \vec{p}')\, \delta n(\vec{p}', \vec{r})\, \frac{}{(2\pi\hbar)^3}$$

and $\dfrac{\partial n_0}{\partial \vec{p}} = \dfrac{\partial n_0}{\partial \varepsilon}\vec{v}$, $\dfrac{\partial \delta n}{\partial \vec{r}} \to \vec{k}$

The above equation is an equation on $\omega(\vec{k})$

Types of solutions

Collective

Quasiparticles

$$\omega = \vec{k}\vec{v}$$

$$u(\vec{n}') \propto \delta(\vec{n} - \vec{n}') + u_{reg}(\vec{n}')$$

To find collective modes, consider $F(\vec{n}, \vec{n}') = F_0$

Assuming u depends only on the angle,

$$(s - \cos\theta)\, u(\theta) = \frac{1}{2} F_0 \cos\theta \int u(\theta')\, d\theta' \qquad (1)$$

where $s = \dfrac{\omega}{v_F k}$.

The solution is $u(\vec{n}) = \dfrac{A\cos\theta}{s - \cos\theta}$, A - arb. constant

Then from (1) $\dfrac{s}{2} \ln\dfrac{s+1}{s-1} = \dfrac{1}{F_0} + 1$

- has one solution $\forall F_0 > 0$

$$\omega_0(\vec{k}) = s(F_0)\, v_F |\vec{k}|$$

- the dispersion law of zero sound